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A GROUP ACTION ON PICTURE FUZZY SOFT G-MODULES

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ABSTRACT

The target of this study is to observe some of the algebraic structures of a picture uncertainty soft set. So, in this paper we introduce the idea of a picture uncertainty soft G-module for a given classical module and investigate some of the crucial properties as well as properties of the expected concept. The ideas of G-invariant picture uncertainty soft G-modules are also discussed.

KEYWORDS: *Group Action, Classical G-Modules, Fuzzy Set, Soft Set, Picture Fuzzy Uncertainty Soft G-Modules, G-Invariant, Cartesian Product and Intersection*

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INTRODUCTION

The concept of uncertaintysets was first introduced by Zadeh [20] in 1965 and the uncertainty sets have been used in the reconsideration of classical mathematics. Recently, Yuan [19] introduced the concept of uncertainty sub division group with thresholds. A fuzzy subgroup which thresholds "λ and μ", is also called a (λ,μ)–uncertainty subgroups. Yao [16] continued the research on (λ,μ)–fuzzy normal subgroups, (λ,μ)–uncertainty quotient subgroups and (λ,μ)–uncertainty subrings [17]. L.A.Zadeh [20] in 1965 introduced the concept of uncertaintysets to describe vagueness mathematically in its very abstractness. The notion of L- uncertainty sets in lattice theory introduced the concepts of uncertainty sub lattices and uncertainty ideals in [15]. In particular, N. Ajmal and K.V. Thomas [2–4] systematically developed the theory of uncertainty sub lattice. The theory of G-modules originated in the 20th century. Representation theory was developed on the basis of embedding a group G in to a linear group GL (V). The theory of group representation (G module theory) was developed by Frobenious G [1962], soon after the concept of fuzzy sets were introduced by Zadeh in 1965. Uncertainty subgroup and its important properties were defined and established by Rosenfeld in 1971. After that, in the year 2004, Shery Fernandez introduced uncertainty parallels of the notions of G-modules. Here, we give some basic definitions and results related to fuzzy sub lattice which is taken from their work. The concept of group actions are in various algebraic structures in [9, 13]. Let X is a non-empty set. A mapping $\mu: X \rightarrow [0, 1]$ is called a uncertainty subset of X. Rosenfeld [12] applied the concept of uncertainty sets to the theory of groups and defined the concept of uncertainty subgroups of a group. Since then, many papers concerning various uncertainty algebraic structures have appeared in the literature [1, 5–7, 11, 18]. The target of this study is to observe some of the algebraic structures of a picture uncertainty soft set. So, we introduce the concept of a picture uncertainty soft G-module of a given classical module and investigate some of the crucial properties and characterizations of the expected concept. The ideas of G-invariant picture uncertaintysoft G-modules are also discussed in this article.

2. PRELIMINARIES

Let 'M' be a module over the ring of integers Z and G be a countable group which acts on M

((i e) $\forall\ g \in G,\ x \in M,\ x^g = gxg^{-1} \in M$).

The identity element of G is denoted by "e".

Definition 2.1: A group action of G on a uncertainty soft set 'A' of a Z-module M is denoted by A^g and is defined by $A^g(x) = A(x^g),\ g \in G.$

From the definition of group action G on auncertainty soft set, following results are easy to verify.

Definition 2.2: A uncertaintyset μ in a universe X is a mapping μ : X → [0,1].

Definition 2.3: Let G be a finite group. A vector space M over a field K (a subfield of C) is called a G-module if for every g ∈ G and m ∈ M, there exists a multiplication (called the right action of G on M) m.g∈ M which fulfils the following axioms.

- m.1G = m for all m ∈ M (1G being the identify of G)
- m. (g. h) = (m.g). h, m ∈ M, g, h ∈ G
- $(k_1\ m_1 + k_2\ m_2)$. G = $k_1\ (m_1.\ g) + k_2(m_2.\ g)$, k_1, k_2∈ K, m_1, m_2∈ M & g ∈ G.

In a similar manner, the left action of G on M can be defined.

Definition 2.4: Let M and M* be G-modules. A mapping Ø: M→M* is a G-module homomorphism if

- Ø(k1 m1 + k2m2) = k1 Ø (m1) + k2 Ø (m2)
- Ø(gm) = g Ø (m), k1, k2∈K, m, m1, m2∈M & g ∈G.

Definition 2.5: Let 'M' be a G-module. A subspace N of M is a G -sub module if 'N' is also a G-module under the action of G.

Definition 2.6: Let 'U' be any Universal set, E set of parameters and A⊆E. Then a pair (K, A) is called soft set over U, where K is a mapping from A to 2^U, the power set of U.

Example 2.7: Let X={c_1,c_2,c_3} be the set of three cars and E={costly(e_1), metallic color(e_2), cheap(e_3)} be the set of parameters, where A = {e_1,e_2} ⊂ E. Then (K, A) = {K(e_1) = {c_1,c_2,c_3}, K(e_2) = {c_1,c_2}} is the crisp soft set over X.

Definition 2.8: Let 'U' be an initial universe. Let P(U) be the power set of U, E be the collection of all parameters and A⊆E. A soft set (f_A, E) on the universe U is defined by the set of order pairs (f_A, E)={(e, f_A(e)):e∈E, f_A∈P(U)} where f_A:E⟶P(U) such that f_A (e) = ϕ, f (e) ∉A. Here 'f_A' is called an approximate function of the soft set.

Example 2.9: Let U = {u_1,u_2, u_3, u_4} be a set of four shirts and E = {white (e_1), red (e_2), blue (e_3)} be a set of parameters. If A={e_1, e_2}⊆E. Let $f_A(e_1)$ ={u_1,u_2,u_3,u_4} and $f_A(e_2)$ = {u_1,u_2,u_3}. Then we know the soft set (f_A, E) = {(e_1, {u_1, u_2,u_3, u_4}), (e_2, {u_1,u_2,u_3})} over U which explain the "color of the shirts" which Mr. X is going to buy. We may represent the soft set in the following form: U = {(e_1,u_1), (e_2,u_1), (e_1,u_2), (e_2,u_2), (e_1,u_3), (e_2,u_3), (e_1, u_4)}.

Definition 2.10: Let 'U' be the universal set, E set of parameters and A⊂ E. Let K(X) denote the set of all uncertainty subsets of U. Then a pair (K, A) is called uncertainty soft set over U, where 'K' is a mapping from A to K(U).

Example 2.11: Let U = {c_1, c_2, c_3} be the set of three cars and E = {costly (e_1), metallic color (e_2), cheap (e_3)} be the collection of parameters, where A={e_1,e_2}⊂E. Then (K,A) = {K(e_1) = {c_1/0.6, c_2/0.4, c_3/0.3}, K(e_2) = {c_1/0.5, c_2/0.7, c_3/0.8}} is the uncertainty soft set over U denoted by F_A.

Definition 2.12: Let K_A be a uncertainty soft set over U and 'α' be a subset of U, then upper α- inclusion of K_A denoted by K^{α}_A={x∈ A/K(x)≥α}. Similarly K^{α}_A={x∈ A/K(x)≤α} is called lower α-inclusion of K_A.

Definition 2.13: Let K_A and G_B be uncertainty soft sets over the common universe U and ψ:A⟶ B be a function. Then, uncertainty soft image of K_A under ψ over U denoted by ψ(K_A) is a set-valued function, where ψ(K_A) : B⟶ 2^U defined by ψ(K_A) (b) = {∪{K(a) / a∈ Aand ψ (a)=b},if ψ^{-1}(b)≠ φ} for all b∈ B, the soft pre-image of G_B under ψ over U denoted by ψ^{-1}(G_B) is a set-valued function, whereψ^{-1}(G_B) : A → 2^U defined by ψ^{-1}(G_B)(b) = G(ψ(a)) for all a ∈ A. Then uncertainty soft anti-image of K_A under ψ over U denoted by ψ(K_A) is a set-valued function, where ψ(K_A):B→2^U defined by ψ^{-1}(K_A)(b)={∩{K(a)/a∈ A and ψ(a)=b}, if ψ^{-1}(b) ≠ φ, for all b∈ B.

Lemma 2.14 [P.K.Sharma]: Let 'G' be a countable group which acts on Z-module M. Then for every $x, y \in M,\ g \in G \text{ and } r \in Z,$ we get,

- $(x+y)^g = x^g + y^g$
- $(xy)^g = x^g y^g$
- $(rx)^g = r\,x^g$
- $(x, y)^g = (x^g, y^g)$

Proof: (i) Since $(x+y)^g = g(x+y)g^{-1}$

$$= gxg^{-1} + gyg^{-1}$$

$$= x^g + y^g$$

(ii) $(xy)^g = g(xy)g^{-1} = g(xey)g^{-1}$

$$= g(xg^{-1}gy)g^{-1}$$

$$= (gxg^{-1})(gyg^{-1})$$

$$= x^g y^g$$

(iii) $(rx)^g = g(rx)g^{-1}$

$$= g(x + x + \dots\dots + r\ times)g^{-1}$$

$$= gxg^{-1} + gxg^{-1} + gxg^{-1} + \dots\dots + r\ times$$

$$= r\left(gxg^{-1}\right)$$

$$= r\ x^{g}$$

(iv) $(x, y)^{g} = g(x, y)g^{-1}$

$$= \left(gxg^{-1}, gyg^{-1}\right)$$

$$= \left(x^{g}, y^{g}\right).$$

3. PICTURE UNCERTAINTY SOFT G-MODULES

In this part, we define the concept of Picture Fuzzy Soft G-module of a given classical module over a ring and also study its elementary properties and characterizations. Throughout this article, R denotes a commutative ring with unity 1.

Definition 3.1: Let 'M' be a module over a ring R. A picture uncertainty soft set 'A' on M is called a picture uncertaintysoft G-module of M if the following conditions are satisfied:

(PFSGM–1): $A(0) = X$

(ie) $P_A(0)=1, \quad N_A(0)=1, \quad I_A(0)=0$

(PFSGM–2): $A(x+y) \geq Min\{A(x), A(y)\}, \ for\, each\ x, y \in M$

(ie) $P_A(x+y) \geq Min\{P_A(x), P_A(y)\}$,

$N_A(x+y) \geq Min\{N_A(x),\ N_A(y)\}$,

$I_A(x+y) \leq Max\{I_A(x),\ I_A(y)\}$.

(PFSGM–3) : $A(rx) \geq A(x), \ \ for\ \ each\ \ x \in M,\ r \in R,$

(ie) $P_A(rx) \geq P_A(x)$,

$N_A(rx) \geq N_A(x)$,

$I_A(rx) \leq I_A(x)$.

The collection of all picture uncertainty soft G-modules of M is denoted by PFSG(M).

Example 3.2: Let us take the classical ring R = Z_4 = {0, 1, 2, 3}. Since each ring is a module on itself, we consider M = Z_4 as a classical module.

Define a picture uncertainty soft set 'A' as follows

$$A=\{(1,1,0)/0+(0.7,0.4,0.7)/1+(0.8,0.2,0.4)/2+(0.7,0.2,0.7)/3\}$$

It is clear that the picture uncertaintysoft set 'A' is a picture uncertaintysoft G-module of M.

Proposition 3.3: Let 'A' be a picture uncertainty soft set of Z-module M and G be a finite group which acts on M. The A^g is also a picture uncertainty soft G-module of M.

Proof: Clearly, (PFSGM–1): $A^g(0)=A(0^g)=A(0)=1.$

(PFSGM–2): Let $x, y\in M,\ g\in G\ and\ r\in Z$, then by Lemma 2.14 (i),

$$A^g(x+y)=A\{(x+y)^g\}$$

$$=A(x^g+y^g)$$

$$\geq \min\{A(x^g), A(y^g)\}$$

$$=\min\{A^g(x),\ A^g(y)\}$$

(PFSGM–3): $A^g(rx)=A\{(rx)^g\}$

$$=A(r\,x^g)$$

$$\geq A(x^g)$$

$$=A^g(x)$$

By Lemma 2.14 (i) and (iii)

Hence, A^g is picture uncertaintysoft G-module of M.

Remark 3.4: The revers of Proposition 3.3 does not hold.

Example 3.5: Let $M=\{Z_4=\{0,1,2,3\},+4, X_4\}$ regarded as Z-module and a finite group $G=(\{0,1,2,3,4\}, X_5)$.

Consider a picture uncertainty soft set A of M given by $A(0)=0.2,\ A(1)=0.3,\ A(2)=0.7,\ A(3)=0.4.$

Clearly, 'A' is not picture uncertainty soft G-module of M, because

$$A(2+4^3)=A(1)=0.3<0.4$$

$$=\min\{0.7,\ 0.4\}$$

$$=\min\{A(2),\ A(3)\}$$

Take g = 3, so that $g^{-1}=2$, then

$$x^g = gxg^{-1} = 3X_4 \; x \; X_4 \; 2 = 6x(\text{mod } 4) = 2x(\text{mod } 4), \text{ We get } A^g(x) = A(x^g) = \begin{cases} 1, & if \;\; x = 0,\, 2 \\ 0.4, & if \;\; x = 1,\, 3 \end{cases}$$

Now, it is easy to check that A^g is picture uncertainty soft G-module of M.

Definition 3.6: Let A and B be picture uncertainty soft sets on M, then their sum A+B is a picture uncertainty soft set on M, defined as follows:

$$P_{A+B}(x) = \max\{\min\{P_A(y), P_B(z)\} / x = y + z, y, z \in M\}$$

$$N_{A+B}(x) = \max\{\min\{N_A(y), N_B(z)\} / x = y + z, y, z \in M\}$$

$$I_{A+B}(x) = \min\{\max\{I_A(y), I_B(z)\} / x = y + z, y, z \in M\}$$

Definition 3.7: Let 'A' be a picture uncertainty soft sets on M, then – A is a picture uncertainty soft set on M, defined as follows:

$$P_{-A}(x) = P_A(-x), N_{-A}(x) = N_A(-x), I_{-A}(x) = I_A(-x), \;\; for \; each \;\; x \in M$$

4. CHARACTERIZATION OF PICTURE UNCERTAINTY SOFT G-MODULES

In this part, we analyse the structure of picture uncertainty soft G-modules under group actions. The following theorems are proved enhanced with group actions.

Definition 4.1: Let 'A' be a picture uncertainty soft sets on M and r∈ R. Define picture uncertainty soft set rA on M as follows:

$$P_{rA} = \max\{P_A(y) / y \in M, \;\; x = ry\}$$

$$N_{rA} = \max\{N_A(y) / y \in M, \;\; x = ry\} \; and$$

$$I_{rA} = \min\{I_A(y) / y \in M, \;\; x = ry\}.$$

Proposition 4.2: If 'A' is a picture uncertainty soft set of G-module of M and G be a finite group which acts on M, then (–1)A = –A.

Proof: Let x∈ M be arbitrary.

$$P_{(-1)A}(x^g) = \bigvee_{x=(-1)y} P_A(y^g)$$

$$= \bigvee_{y=-x} P_A(x^g)$$

$$= P_A(-x^g)$$

$$= P_{-A}(x^g)$$

Similarly, $N_{(-1)A}\left(x^{g}\right)=N_{-A}\left(x^{g}\right)$ *and*

$I_{(-1)A}\left(x^{g}\right)=I_{-A}\left(x^{g}\right),\ \ for\ each\ \ x\in M$

Then the following is valid.

$$(-1)A=\left(P_{(-1)A},\ N_{(-1)A},\ I_{(-1)A}\right)=\left(P_{-A},N_{-A},\ I_{-A}\right)=-A$$

This completes the proof.

Proposition 4.3: If ‘A’ and ‘B’ are picture uncertainty soft sets on M, with $A\subseteq B,\ then\ rA\subseteq rB,$ for each r∈ R.

Proof: It is straightforward by the definition.

Proposition 4.4: If ‘A’ and ‘B’ are picture uncertaintysoft sets on M and G be a countable group which acts on M, then $r(sA)=(rs)A,$ for each r, s∈ R.

Proof: Let x∈ M and r, s∈ R be arbitrary.

$$I_{r(sA)}\left(x^{g}\right)=\bigwedge_{x=ry} I_{sA}\left(y^{g}\right)$$

$$=\bigwedge_{x=ry}\ \bigwedge_{y=sx} I_{A}\left(z^{g}\right)$$

$$=\bigwedge_{x=r(sz)} I_{A}\left(z^{g}\right)$$

$$=\bigwedge_{x=(rs)z} I_{A}\left(z^{g}\right)$$

$$=I_{(rs)A}\left(x^{g}\right)$$

By the routine calculations the other equalities are obtained, so

$$r(sA)=\left(P_{r(sA)},\ N_{r(sA)},\ I_{r(sA)}\right)$$

$$=\left(P_{(rs)A},\ N_{(rs)A},\ I_{(rs)A}\right)$$

$$=(rs)A$$

Hence the proof.

Proposition 4.5: If ‘A’ and ‘B’ are picture uncertainty soft sets on M and G be a countable group which acts on M, then $r(A+B)=rA+rB$ for each r∈ R.

Proof: Let ‘A’ and ‘B’ are picture uncertainty soft sets on M, x∈ M and r∈ R.

$$N_{r(A+B)}\left(x^{g}\right)=\bigvee_{x=ry} N_{A+B}\left(y^{g}\right)$$

$$= \bigvee_{x=ry} \bigvee_{y=x_1+x_2} \min\{N_A(z_1^g), N_B(z_2^g)\}$$

$$= \bigvee_{x=rx_1+rx_2} \min\{N_A(z_1^g), N_B(z_2^g)\}$$

$$= \bigvee_{x=x_1+x_2} \min\left\{\left(\left(\bigvee_{x_1=rz_1} N_A(z_1^g)\right), \left(\bigvee_{x_2=rz_2} N_B(z_2^g)\right)\right)\right\}$$

$$= \bigvee_{x=x_1+x_2} \min\{N_{rA}(x_1^g), N_{rB}(x_2^g)\}$$

$$= N_{rA+rB}(x^g).$$

The other equalities are obtained similarly.

Hence, $r(A+B) = (P_{r(A+B)}, N_{r(A+B)}, I_{r(A+B)})$

$$= (P_{rA+rB}, N_{rA+rB}, I_{rA+rB})$$

$$= rA + rB.$$

Proposition 4.6: If ‘A’ is picture uncertaintysoft sets on M and G be a countable group which acts on M, then $P_{rA}(rx)^g \geq P_A(x^g), N_{rA}(rx)^g \geq N_A(x^g), I_{rA}(rx)^g \leq I_A(x^g).$

Proof: It is straightforward by the definition.

Proposition 4.7: If ‘A’ and ‘B’ are picture uncertainty soft sets on M and G be a countable group which acts on M, then

$$P_B(rx)^g \geq P_A(x^g), \quad \text{for each } x \in M,$$

$$\text{if and only if } P_{rA} \leq P_B$$

$$N_B(rx)^g \geq N_A(x^g), \quad \text{for each } x \in M,$$

$$\text{if and only if } N_{rA} \leq N_B$$

$$I_B(rx)^g \leq I_A(x^g), \quad \text{for each } x \in M,$$

$$\text{if and only if } I_{rA} \geq I_B$$

Proof: (i) Suppose $P_B(rx)^g \geq P_A(x^g)$, $\text{for each } x \in M$,

then $P_{rA}(x^g) = \bigvee_{x=ry,\ y \in M} P_A(y^g)$,

So, $P_{rA} \leq P_B$.

Conversely, suppose $P_{rA} \leq P_B$ is satisfied,

then $P_{rA}\left(x^g\right) \leq P_B\left(x^g\right),\ \ for\ \ each\ \ x \in M.$

Hence, $P_B\left(x^g\right) \geq P_{rA}(rx)^g \geq P_A\left(x^g\right),\ \ for\, each\ \ x \in M$

by Proposition 4.6), (ii) and (iii) are proved in a similar way.

Proposition 4.8: If ‘A’ and ‘B’ are picture uncertaintysoft sets on M and G be a countable group which acts on M, then

- $P_{rAsB}(rx+sy)^g \geq \min\left\{P_A\left(x^g\right), P_B\left(y^g\right)\right\}$
- $N_{rAsB}(rx+sy)^g \geq \min\left\{N_A\left(x^g\right), N_B\left(y^g\right)\right\}$
- $I_{rAsB}(rx+sy)^g \leq \max\left\{I_A\left(x^g\right), I_B\left(y^g\right)\right\},\ \ for\ \ each\ \ x \in M,\ r, s \in R.$

Proof: It is proved by using Definition 3.6, Definition 3.8 and Proposition 4.6.

Proposition 4.9: If A, B and C are three picture uncertainty soft sets on M and ‘G’ be a countable group which acts on M, then the following are satisfied for each r, s ∈ R:

- $P_C(rx+sy)^g \geq \min\left\{P_A\left(x^g\right), P_B\left(y^g\right)\right\},\ for\, all\ x, y \in M\ if\ and\ only\, if\ \ P_{rA+sB} \leq P_C$
- $N_C(rx+sy)^g \geq \min\left\{N_A\left(x^g\right), N_B\left(y^g\right)\right\},\ \ for\, all\ \ x, y \in M\ if\ and\ only\, if\ N_{rA+sB} \leq N_C$
- $I_C(rx+sy)^g \leq \max\left\{I_A\left(x^g\right), I_B\left(y^g\right)\right\},\ \ for\, all\ \ x, y \in M\ \ if\ and\ only\, if\ \ I_{rA+sB} \geq I_C$

Proof: It is proved by using Proposition 4.8.

Theorem 4.10: Let ‘A’ be a picture uncertainty soft sets on M and ‘G’ be a countable group which acts on M and for each r, s ∈ R, then

$$P_{rA} \leq P_A \iff P_A(rx)^g \geq P_A\left(x^g\right),$$

$$N_{rA} \leq N_A \iff N_A(rx)^g \geq N_A\left(x^g\right)\ \ and$$

$$I_{rA} \geq I_A \iff I_A(rx)^g \leq I_A\left(x^g\right)\, for\, each\, x \in M.$$

$$P_{rA+sA} \leq P_A \iff P_A(rx+sy)^g \geq \min\left\{P_A\left(x^g\right), P_A\left(y^g\right)\right\}$$

$$N_{rA+sA} \leq N_A \iff N_A(rx+sy)^g \geq \min\left\{N_A\left(x^g\right), N_A\left(y^g\right)\right\}$$

$$I_{rA+sA} \geq I_A \iff I_A(rx+sy)^g \leq \max\left\{I_A\left(x^g\right), I_A\left(y^g\right)\right\}$$

Proof: The Proof follows from proposition 4.7 and Proposition 4.9.

Theorem 4.11: Let ‘A’ be a picture uncertainty soft sets on M and ‘G’ be a countable group which acts on M.Then A∈ PFSGM(M) if and only if the following properties are fulfiled.

- $A(0^g) = \tilde{X}$.
- $A(rx+sy)^g \geq \min\{A(x^g),\ A(y^g)\}$, .

  $for\ each\ x, y \in M\ and\ r, s \in R$

Proof: Let 'A' be a picture uncertainty soft G-module on M and 'G' be a countable group which acts on M and x, y ∈ M.

From the axiom (PFSGM–1) of Definition 3.1, it is clearly that $A(0^g) = \tilde{X}$.

From (PFSGM–2) and (PFSGM–3), the following are true,

$$P_A(rx+sy)^g \geq \min\{P_A(rx^g),\ P_A(sy^g)\}$$

$$\geq \min\{P_A(x^g),\ P_A(y^g)\}$$

$$N_A(rx+sy)^g \geq \min\{N_A(rx)^g,\ N_A(sy)^g\}$$

$$\geq \min\{N_A(x^g),\ N_A(y^g)\}\ \ and$$

$$I_A(rx+sy)^g \leq \max\{I_A(rx)^g,\ I_A(sy)^g\}$$

$$\leq \max\{I_A(x^g),\ I_A(y^g)\},\quad for\ each\ x, y \in M\ \ and\ \ r, s \in R$$

Hence $A(rx+sy)^g = \left(P_A(rx+sy)^g, N_A(rx+sy)^g, I_A(rx+sy)^g\right)$

$$\geq \begin{pmatrix} \min\{P_A(x^g),\ P_A(y^g)\}, \\ \min\{N_A(x^g),\ N_A(y^g)\}, \\ \min\{I_A(x^g),\ I_A(y^g)\} \end{pmatrix}$$

$$= \begin{pmatrix} \min\{P_A(x^g),\ N_A(x^g),\ I_A(x^g)\}, \\ \min\{P_A(y^g),\ N_A(y^g),\ I_A(y^g)\} \end{pmatrix}$$

$$= \min\{A(x^g),\ A(y^g)\}.$$

Conversely, suppose 'A' exists the condition (i) and (ii) then it is clearly hypothesis $A(0^g) = \tilde{X}$.

$$P_A(x+y)^g = P_A(1.x+1.y)^g \geq \min\{P_A(x^g), P_A(y^g)\}$$

$$N_A(x+y)^g = N_A(1.x+1.y)^g \geq \min\{N_A(x^g), N_A(y^g)\}$$

$$I_A(x+y)^g = I_A(1.x+1.y)^g \leq \max\{I_A(x^g), I_A(y^g)\}$$

So, $A(x+y)^g \geq \min\{A(x^g),\ A(y^g)\}$ and the conditions (PFSGM–2) of Definition 3.1 is satisfied.

Now, let us show the validity of condition (PFSGM–3), by the hypothesis,

$$P_A(rx)^g = P_A(rx + r0)^g \geq \min\{P_A(x^g), P_A(0^g)\} = P_A(x^g)$$

$$N_A(rx)^g = N_A(rx + r0)^g \geq \min\{N_A(x^g), N_A(0^g)\} = N_A(x^g) \; I_A(rx)^g = I_A(rx + r0)^g \leq \max\{I_A(x^g), I_A(0^g)\} = I_A(x^g)$$

for each $x, y \in M$, $r \in R$.

Therefore, (PFSGM–3) of Definition 3.1 is satisfied.

Theorem 4.12: Let 'A' and 'B' are picture uncertaintysoft G-modules of a classical module M and G be a countable group which acts on M.Then intersection A $\cap$ B is also a PFSGM of M.

Proof: Since A, B $\in$ PFSGM(M), we have

$$A(0^g) = \tilde{X}, \quad B(0^g) = \tilde{X}.$$

$$P_{A\cap B}(0^g) = \min\{P_A(0^g), P_B(0^g)\} = 1$$

$$N_{A\cap B}(0^g) = \min\{N_A(0^g), N_B(0^g)\} = 1$$

$$I_{A\cap B}(0^g) = \max\{I_A(0^g), I_B(0^g)\} = 0.$$

Let $x, y \in M$, $r, s \in R$, by Theorem 4.11, it is enough to show that

$$(A \cap B)(rx + sy)^g \geq \min\{(A \cap B)(x^g), (A \cap B)(y^g)\}$$

$$P_{A\cap B}(rx + sy)^g \geq \min\{P_{A\cap B}(x^g), P_{A\cap B}(y^g)\}$$

$$N_{A\cap B}(rx + sy)^g \geq \min\{N_{A\cap B}(x^g), N_{A\cap B}(y^g)\}$$

$$I_{A\cap B}(rx + sy)^g \leq \max\{I_{A\cap B}(x^g), I_{A\cap B}(y^g)\}$$

Now, we consider the truth-membership degree of the intersection,

$$P_{A\cap B}(rx + sy)^g = \min\{P_A(rx + sy)^g, P_B(rx + sy)^g\}$$

$$\geq \min\{\min\{P_A(x^g), P_A(y^g)\}, \min\{P_B(x^g), P_B(y^g)\}\}$$

$$= \min\{\min\{P_A(x^g), P_B(x^g)\}, \min\{P_A(y^g), P_B(y^g)\}\}$$

$$= \min\{P_{A\cap B}(x^g), P_{A\cap B}(y^g)\}$$

Then, other inequalities are proved similarly.

Hence, A $\cap$ B $\in$ PFSGM (M).

Note 4.13: A non-empty subset N of M is a sub module of M if and only if rx+sy$\in$ N for all $x, y \in M$, $r, s \in R$,

Proposition 4.14: Let 'M' be a G-module over R. A$\in$ PFSGM(M) and G be a countable group which acts on M, if and only if for all $\alpha \in [0, 1]$, $\alpha - level\, sets\, of\; A$, $(P_A)_\alpha, (N_A)_\alpha, (I_A)^\alpha$ are classical G-modules of M where

$A(0^g) = \tilde{X}$.

Proof: Let A∈ PFSGM(M), $\alpha \in [0, 1], x, y \in (P_A)_\alpha. \ and \ r, s \in R \, be \, any \, elements.$

Then, $P_A(x^g) \geq \alpha, \ P_A(y^g) \geq \alpha$ and $\min\{P_A(x^g), P_A(y^g)\} \geq \alpha$.

By using Theorem 4.11, we have,

$P_A(rx+sy)^g \geq \min\{P_A(x)^g, \ P_A(y)^g\} \geq \alpha.$

Hence, $rx + sy \in (P_A)_\alpha$

Therefore, $(P_A)_\alpha$ is a classical G-module of M for each $\alpha \in [0, 1]$.

Similarly, for $x, y \in (N_A)_\alpha, \ (I_A)^\alpha$

We obtain $rx + sy \in (N_A)_\alpha, \ (I_A)^\alpha$ for each $\alpha \in [0, 1]$.

Consequently, $(N_A)_\alpha, \ (I_A)$ are classical G-module of M for each $\alpha \in [0, 1]$.

Conversely, let $(P_A)_\alpha$ be a classical G-module of M for each $\alpha \in [0, 1]$.

Let $x, y \in M, \quad \alpha = \min\{P_A(x)^g, \ P_A(y)^g\}$ then

$P_A(x)^g \geq \alpha \ and \ P_A(y)^g \geq \alpha..$

Thus, $x, y \in (P_A)_\alpha$.

Since, $(P_A)_\alpha$ is a classical G-module of M, we have

$rx + sy \in (P_A)_\alpha$ for all $r, s \in R$.

Hence, $(P_A)(rx+sy)^g \geq \alpha = \min\{P_A(x^g), \ P_A(y^g)\}.$

Similarly, we obtain

$(N_A)(rx+sy)^g \geq \min\{N_A(x^g), \ N_A(y^g)\}.$

Now, we consider $(I_A)^\alpha$,

$let \quad x, y \in M, \quad \alpha = \max\{I_A(x^g), I_A(y^g)\} \ then$

$I_A(x^g) \leq \alpha, \quad I_A(y^g) \leq \alpha..$

Thus, $x, y \in (I_A)^\alpha$.

Since, $(I_A)^\alpha$ is a G-module of M, we have $rx+sy\in(I_A)^\alpha$ for all $r,s\in R$.

Thus, $(I_A)(rx+sy)^g \le \alpha = \max\{I_A(x^g),\ I_A(y^g)\}$.

It is also obvious that $A(0^g) = \tilde{X}$.

Hence, the conditions of Theorem 4.11 are satisfied.

Proposition 4.15: Let ‘A’ and ‘B’ are two picture uncertainty soft sets X and Y respectively and G be a countable group which acts on M. Then the following equalities are satisfied for the α-levels $(P_{A\times B})_\alpha = (P_A)_\alpha \times (P_B)_\alpha$,

$$(N_{A\times B})_\alpha = (N_A)_\alpha \times (N_B)_\alpha$$

$$(I_{A\times B})^\alpha = (I_A)^\alpha \times (P_B)^\alpha.$$

Proof: Let $(x,y) = (P_{A\times B})_\alpha$ be arbitrary.

So, $P_{A\times B}(x,y)^g \ge \alpha \Leftrightarrow \min\{P_A(x^g), P_B(y^g)\} \ge \alpha$

$\Leftrightarrow P_A(x^g) \ge \alpha \text{ and } P_B(y^g) \ge \alpha$

$\Leftrightarrow (x,y)^g \in (P_A)_\alpha \times (P_B)_\alpha$

$N_{A\times B}(x,y)^g = (N_A)_\alpha \times (N_B)_\alpha$ is proved in a similarly way.

Let $(x,y) = (I_{A\times B})^\alpha$ be arbitrary.

Hence, $I_{A\times B}(x,y)^g \le \alpha \Leftrightarrow \max\{I_A(x^g), I_B(y^g)\} \le \alpha$

$\Leftrightarrow I_A(x^g) \le \alpha \text{ and } I_B(y^g) \le \alpha$

$\Leftrightarrow (x,y)^g \in (I_A)^\alpha \times (I_B)^\alpha.$

Proposition 4.16: Let A, B∈ PFSGM(M).Then the product A×B is also a picture uncertainty soft G-module of M.

Proof: We know that the direct product of two soft G-modules is a G-module. So, by proposition 4.14 and proposition 4.15, we obtain the result.

Proposition 4.17: Let ‘A’ and ‘B’ are two pictureuncertainty fuzzy soft sets on X and Y respectively and G be a countable group which acts on M and $\phi: X \to Y$ be a mapping. Then the followings hold:

(i) $\phi((P_A)_\alpha) \subseteq (P_{\phi(A)})_\alpha$,

$\phi((N_A)_\alpha) \subseteq (N_{\phi(A)})_\alpha$,

$\phi((I_A)^\alpha) \supseteq (I_{\phi(A)})^\alpha$.

(ii) $\phi^{-1}\left((P_B)_\alpha\right) = \left(P_{\phi^{-1}(B)}\right)_\alpha$,

$\phi^{-1}\left((N_B)_\alpha\right) = \left(N_{\phi^{-1}(B)}\right)_\alpha$,

$\phi^{-1}\left((I_B)^\alpha\right) = \left(I_{\phi^{-1}(B)}\right)^\alpha$.

Proof: (i) Let $y \in \phi\left((P_A)_\alpha\right)$, then there exists $x \in (P_A)_\alpha$ such that $\phi(x) = y$.

Hence, $P_A(x^g) \geq \alpha$.

So, $\bigvee_{x \in \phi^{-1}(y)} P_A(x) \geq \alpha$,

(ie) $P_{\phi(A)}(y^g) \geq \alpha$ *and* $y \in \left(P_{\phi(A)}\right)_\alpha$

Hence, $\phi\left((P_A)_\alpha\right) \subseteq \left(P_{\phi(A)}\right)_\alpha$, similarly, we obtain other inclusions.

(ii) $\left(P_{\phi^{-1}(B)}\right)_\alpha = \left\{x \in X / P_{\phi^{-1}(B)}(x^g) \geq \alpha\right\}$

$= \left\{x \in X / P_B\phi(x^g) \geq \alpha\right\}$

$= \left\{x \in X / \phi(x) \in (P_B)_\alpha\right\}$

$= \left\{x \in X / x \in \phi^{-1}\left((P_B)_\alpha\right)\right\}$

$= \phi^{-1}\left((P_B)_\alpha\right)$.

The other equalities are obtained in a similar way.

Theorem 4.18: Let M, N be the classical G-modules and $\phi : M \to N$ be a homomorphism of pre image $\phi^{-1}(B)$ is a PFSGM of M.

Proof: By proposition 4.17 (ii), we have

$\phi^{-1}\left((P_B)_\alpha\right) = \left(P_{\phi^{-1}(B)}\right)_\alpha$,

$\phi^{-1}\left((N_B)_\alpha\right) = \left(N_{\phi^{-1}(B)}\right)_\alpha$,

$\phi^{-1}\left((I_B)^\alpha\right) = \left(I_{\phi^{-1}(B)}\right)^\alpha$.

Since Pre image of a G-module is a G-module, by proposition 4.14, we obtain the result.

Corollary 4.20: If $\phi : M \to N$ is a homomorphism of G-module and $\{B_j : j \in I\}$ is a family of picture fuzzy soft G-modules of N, then the image $\phi^{-1}\left(\bigcap B_j\right)$ is a PFSGM of M.

Theorem 4.21: Let 'M' and 'N' be the classical G-modules and $\phi : M \to N$ be a homomorphism of G-modules. If 'A' is a PFSGM of M and G is a countable group which acts on M, then the image $\phi(A)$ is a PFSGM of N.

Proof: By proposition 4.14, it is enough to show that $\left(P_{\phi(A)}\right)_\alpha, \left(N_{\phi(A)}\right)_\alpha, \left(I_{\phi(A)}\right)_\alpha$ are G-sub modules of N for all $\alpha \in [0, 1]$. Let $y_1, y_2 \in \phi((P_A)_\alpha)$.

Then $P_{\phi(A)}\left(y_1{}^g\right) \geq \alpha$ *and* $P_{\phi(A)}\left(y_2{}^g\right) \geq \alpha$ there exists $x_1, x_2 \in M$ such that

$$P_A\left(x_1{}^g\right) \geq \alpha, \quad P_{\phi(A)}\left(y_1{}^g\right) \geq \alpha \quad and$$

$$P_A\left(x_2{}^g\right) \geq \alpha, \quad P_{\phi(A)}\left(y_2{}^g\right) \geq \alpha.$$

Then $P_A\left(x_1{}^g\right) \geq \alpha, \quad P_A\left(x_2{}^g\right) \geq \alpha$ *and*

$$\min\left\{P_A\left(x_1{}^g\right), P_A\left(x_2{}^g\right)\right\} \geq \alpha.$$

Since 'A' is a PFSGM of M, for any $r, s \in R$, we have

$$P_A(rx_1 + sx_2)^g \geq \min\left\{P_A(x_1)^g, P_A(x_2)^g\right\} \geq \alpha$$

Hence, $rx_1 + sx_2 \in (P_A)_\alpha \Rightarrow \phi(rx_1 + sx_2) \in \phi(P_A)_\alpha \subseteq \left(P_{\phi(A)}\right)$

$$\Rightarrow r\,\phi(x_1) + s\,\phi(x_2) \in \left(P_{\phi(A)}\right)_\alpha$$

$$\Rightarrow ry_1 + sy_2 \in \left(P_{\phi(A)}\right)_\alpha.$$

Therefore, $\left(P_{\phi(A)}\right)_\alpha$ is a G-sub module of N.

Similarly, $\left(N_{\phi(A)}\right)_\alpha, \left(I_{\phi(A)}\right)^\alpha$ are classical G-sub modules of N for each $\alpha \in [0, 1]$.

By Proposition 4.14, $\phi(A)$ is a PFSGM of N.

Corollary 4.22: If $\phi : M \to N$ is a surjective G-module homomorphism and $\{A_i\,; i \in I\}$ is a family of picture uncertainty soft G-modules of M, then the image $\phi^{-1}\left(\bigcap A_i\right)$ is a PFSGM of N.

Definition 4.23: Let 'A' be a PFSGM of M and G be a countable group which acts on M, then 'A' is said to be G-invariant PFSGM of M if and only if $A^g(x) = A\left(x^g\right) \geq A(x), \quad for\ all\ x \in M,\ g \in G.$

Theorem 4.24: Let M and M' be Z-module which G acts and let 'f' be a bijective G-module homomorphism from M, then $f(A)$ is a G-invariant PFSGM of M.

Proof: Since 'A' is a G-invariant PFSGM of M', therefore $A^g = A,\ for\ g \in G.$

Now, $(f(A))^g = f(A^g) = f(A), \forall g \in G.$

Hence, $f(A)$ is G-invariant PFSGM of M'.

CONCLUSIONs

From the philosophical point of set view, it has been shown that a picture uncertaintysoft set generalizes a usual set, fuzzy set, interval valued uncertainty set, intuitionistic uncertainty set etc. A picture uncertainty soft set is an instance of picture uncertaintyy set which can be used in real scientific and E-generating problems.

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